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"A State-Space Approach to Parametrization of Stabilizing Controllers for Nonlinear Systems"

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A State-Space Approach to Parametrization of Stabilizing Controllers for Nonlinear Systems

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Abstract

A state-space approach to Youla-parametrization of stabilizing controllers for linear and nonlinear systems is suggested. The stabilizing controllers (or a class of stabilizing controllers for nonlinear systems) are characterized as (linear/nonlinear) fractional transformations of stable parameters. The main idea behind this approach is to decompose the output feedback stabilization problem into state feedback and state estimation problems. The parametrized output feedback controllers have separation structures. A separation principle follows from the construction. This machinery allows the parametrization of stabilizing controllers to be conducted directly in state space without using coprime-factorization.

Keywords: Fractional Transformation, Input-to-State Stability, Lyapunov Technique, Robust Control, Separation Principle, Stabilization, State Space, Youla-Parametrization

1 Introduction

Yaula-parametrization for linear systems has two properties, i.e., (i) the free parameter set for the parametrized controllers is actually a linear space, and (ii) the stabilizing closed loop maps are also parametrized, and are affine in the free stable parameters. This fact therefore makes it possible to (exactly) solve various robust and optimal control problems (see for example, [43, 9, 44, 10, 41, 13, 2, 6] and references therein). As the basic requirement or constraint for feedback control design is that the designed controllers stabilize the feedback system, while Youla-parametrization provides a systematic way to choose the (optimal) stabilizing controllers. In the Youla-parametrization formula, each input-output (I/O) stabilizing controller can be characterized as a linear fractional transformation of some (I/O) stable parameter. The basic technique used in the derivation is coprime factorization. Due to the clear connections between the stability notions in both the I/O description and the state-space description for a linear system, a state-space formula has also been derived using the coprime factorization technique [10, 25], and each internally stabilizing controller is characterized as a linear fractional transformation of some internally stable system.

When nonlinear systems are considered, it is expected that they could also enjoy the similar controller parametrizations and the properties which the linear parametrized closed-loop systems have. It is indeed the case for a special class of nonlinear systems [8, 7, 12, 37]. However, for a more general class of nonlinear systems, the answer is not very straightforward. As far as the controller parametrization is concerned, a natural approach is to analogically use coprime factorization-like

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technique, although the parametrization formulas for some special cases can be derived without explicitly involving coprime factorization (see, e.g., [8, 7]). Nontheless, the generalizations of the coprime factorization notion for nonlinear systems largely depend on how to define (I/O) stability and coprimeness of (I/O) (stable) operators. There have been a rich variety of versions of coprime factorization for nonlinear systems, because different stability and/or coprimeness notions have been used (*cf.* [17, 37, 29, 31, 35, 5, 26, 38] and references therein). The controller parametrizations can be more or less conducted based on these notions of coprime factorization [17, 37, 35, 26]. However, unlike in the linear system case, neither the computational implications of these results nor their implications in the state space are clear. Some efforts have been made in this direction such that the coprime factorization can be conducted in terms of the state-space techniques [29, 30, 31, 38]. Contrary to the linear systems, one of the difficulties is that the state space stability notion, i.e., asymptotic stability, doesn't imply any I/O stability notion in general. Thence, some concepts, such as the notion of input-to-state (I/S) stability proposed by Sontag in [29, 31], are needed to insure I/O stability by considering asymptotic stability. In particular, in the nice work by Sontag [29, 31], the finite I/O-gain-like stability notion is used to carry out the coprime factorization in the state space, where a finite I/S-gain-like notion of I/S stability is suggested as a bridge between asymptotic stability and I/O stability; it is concluded that if a nonlinear system is smoothly stabilizable, then there is a coprime factorization for the system; moreover, this coprime factorization can be constructed by using smooth state feedback. Verma and Hunt [38] use the similar technique to deal with the coprime factorization in the context of BIBO stability with a slightly different version of coprimeness, and another version of I/S stability, i.e. the bounded-input/bounded-state (BIBS) stability, is used. It is believed that the **potential** use of coprime factorization in the nonlinear control theory is to parametrize the stabilizing compensator laws (see [29]). So there comes up the question: do we really need to use the coprime factorization technique to get the stabilizing controller parametrization?

The answer to the above question is **YES**. In this paper, we derive a parametrization formula of stabilizing controllers for time-invariant linear, input-affine nonlinear, and general nonlinear control systems directly in the state space without using the coprime factorization. We use a state-space technique, which is developed in Doyle et al [11] and is extended in [20] for a more general problem, to deal with the controller parametrization problem. Basically, in this machinery, the general problem is decomposed into some simpler output-estimation and state-feedback problems by a technique of changing variables; the controller parametrization is constructed from the considerations of the simpler problems by the employment of a separation argument. (A separation principle follows from the construction.) In the resulting parametrization formula, the asymptotically stabilizing controllers are characterized as fractional transformations of some asymptotically stable parameters. From the state-space point of view, a parametrized controller is structured as an observer which estimates the state of the plant with zero input, a state feedback which uses the estimated state, and a free stable parameter. In the linear case, this formula is exactly the Youla-parametrization, which characterizes all internally stabilizing time-invariant linear controllers, and the parametrized closed-loop maps are affine in the free parameters. In the nonlinear case, in general, it just characterizes a class of asymptotically stabilizing controllers which have separation structures. This consideration is additionally motivated by some other work in which separation structures for some nonlinear feedback systems are confirmed [39, 28, 4, 33, 23, 18, 36, 1, 21]. Unlike linear systems, the parametrized closed-loop maps do not have similar affine-like representation. In the nonlinear case, the Lyapunov technique is used to deal with stability issue, Sontag's machinery [29, 31, 19] is adopted.

The rest of this paper is organized as follows. In section 2, the linear case is considered to motivate the techniques used in this paper. Parallelly, the input-affine nonlinear system case is considered in section 3, the formula of the parametrized input-affine locally stabilizing controllers is derived. In sections 4, the general nonlinear systems are considered; both local and global formulas are proposed.

Conventions

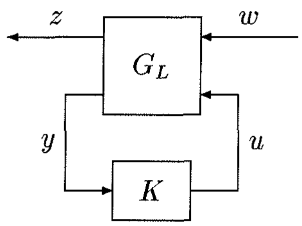
$\mathbb{R}^+ = [0, \infty)$. As in [16], a function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$ is said to be of class $\mathcal{K}$ if it is continuous and strictly increasing; it is of class $\mathcal{K}_\infty$ if in addition $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be of class $\mathcal{KL}$ if for each fixed t, the mapping $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each s, $\beta(s,t)$ is decreasing to 0 as $t \to \infty$. A function is said to be of class $\mathbf{C}^k$ if it is continuously differentiable k times; so $\mathbf{C}^0$ stands for the class of continuous functions. $\mathcal{RH}_\infty$ stands for the class of real rational matrix-valued functions analytic in $\mathbf{Re}(s) > 0$. $\|\cdot\|$ stands for the Euclidean norm of vector in some Euclidean space; $\mathcal{B}_r := \{x \in \mathbb{R}^n : \|x\| < r$ for some integer $n > 0$ and $r > 0$. $\|u\|_\infty :=$ ess-sup$\{\|u(t)\| : t \in \mathbb{R}^+\}$ for $u : \mathbb{R}^+ \to \mathbb{R}^p$. $\mathcal{L}_\infty[0,\infty)$ is the space of functions $u : \mathbb{R}^+ \to \mathbb{R}^p$ which are measurable and essentially bounded; $\mathcal{L}_\infty^e[0,\infty)$ is its extended space [42]. For any $T \geq 0$, P_T denotes the standard truncation operator and $Q_T := I - P_T$. The fractional transformation of M on Q is denoted as as $\mathcal{F}_l(M,Q)$ [27, 15]; and the Redheffer product of M_1 and M_2 is denoted as $\mathcal{S}(M_1, M_2)$ [27].

2 Special Case: Stabilization of Linear Systems

In this section, we consider the linear case and parametrize all stabilizing linear controllers for a linear time-invariant system using the technique by Doyle *et al.* [11], which is detailed in [20] for a more general problem. The aim of this section is to motivate the techniques for the nonlinear stabilization problems.

2.1 Problem Statement

The basic block diagram is as follows:



where G_L is the plant and has the following realization:

$$G_L : \left\{ \begin{array}{rcrcrcr} \dot{x} & = & Ax & + & B_1 w & + & Bu \\ z & = & C_1 x & + & D_{11} w & + & D_{12} u \\ y & = & Cx & + & D_{21} w & + & Du \end{array} \right. \quad \text{or} \quad G_L := \left[\begin{array}{c|cc} A & B_1 & B \\ \hline C_1 & D_{11} & D_{12} \\ C & D_{21} & D \end{array} \right] \tag{1}$$

and K is the controller to be designed. We need to find a linear time-invariant output feedback $K := \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array} \right]$ such that the closed-loop system, denoted as $\mathcal{F}_l(G_L, K)$, is stable.

It is known that a controller stabilizes G_L if and only if it stabilizes $G_0 := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Suppose that G_0 is stabilizable and detectable, then there exist a state-feedback matrix F such that the state feedback system $\dot{x} = (A + BF)x$ is stable, and an output-injection matrix L such that the output injection system $\dot{x} = (A + LC)x$ is stable. It is further assumed that the feedback structure is well-posed, thus $I - D\hat{D}$ is invertible. It is assumed that $D = 0$ without loss of generality [11, 20].

The main results in this section are as follows.

- The stabilizing controller K is parametrized as a linear fractional transformation on some stable parameter Q, i.e., $K = \mathcal{F}_l(J, Q)$ for some linear time-invariant system J, and $Q \in \mathcal{RH}_\infty$.
- The closed loop map from w to z can be represented as $T_{zw} = T_{11} + T_{12}QT_{21}$ for some stable T_{11}, T_{12} and T_{13}.

2.2 Tools: State Feedback and Output Injection

The construction of parametrization given in the next subsection involves the reduction of the original output feedback problem to some simpler problems. These problems are considered to the required extent in this subsection. To this end, we have the following definition.

Definition 2.1 *Two controllers K and K' for system G are equivalent if their corresponding closed loop transfer matrices are identical, i.e. $\mathcal{F}_l(G, K) = \mathcal{F}_l(G, K')$.*

We first examine the stabilization problem when the state x and the input w are fully available to the control. In this case, it is said that the system structure provides full information (FI) (*cf.* [11]). The FI structure is

$$G_{FI} = \left[\begin{array}{c|cc} A & B_1 & B \\ \hline C_1 & D_{11} & D_{12} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right] \tag{2}$$

The stabilizability of G_0 implies that there exists a static feedback matrix F such that $A + BF$ is stable.

Proposition 2.1 *Let F be a constant matrix such that $A + BF$ is stable. Then every stabilizing controller for the FI system (2) is equivalent to one of the controllers in the following parametrized set $\mathcal{K}_{FI}$:*

$$\mathcal{K}_{FI} = \left\{ \begin{bmatrix} F & Q \end{bmatrix} : Q \in \mathcal{RH}_\infty \right\}.$$

Dually, a system structure that the control is directly injected to state and regulated output z is considered. It is called full control (FC) (*cf.* [11]):

$$G_{FC} = \left[\begin{array}{c|c|c} A & B_1 & \begin{bmatrix} I & 0 \end{bmatrix} \\ \hline C_1 & D_{11} & \begin{bmatrix} 0 & I \end{bmatrix} \\ C & D_{21} & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}\right]. \tag{3}$$

Proposition 2.2 *Let L be a constant matrix such that $A+LC$ is stable. Then the set of equivalence classes of all stabilizing controllers for FC system (3) can be parametrized as*

$$\mathcal{K}_{FC} = \left\{ \begin{bmatrix} L \\ Q \end{bmatrix} : Q \in \mathcal{RH}_\infty \right\}.$$

Next, consider a special output feedback structure, which is called output estimation (OE) (*cf.* [11]), defined by

$$G_{OE} = \left[\begin{array}{c|cc} A & B_1 & B \\ \hline C_1 & D_{11} & I \\ C & D_{21} & 0 \end{array} \right]. \tag{4}$$

It is assumed that $A - BC_1$ is stable. OE system is equivalent to FC system in the following sense.

Lemma 2.3 *Consider FC and OE structures (3) and (4). Then (i)* $G_{OE} = G_{FC} \begin{bmatrix} I & 0 \\ 0 & B \\ 0 & I \end{bmatrix}$.

(ii) $G_{FC} = \mathcal{S}(G_{OE}, P_{OE})$, *where* $P_{OE} = \left[\begin{array}{c|c|c} A - BC_1 & 0 & \begin{bmatrix} I & -B \end{bmatrix} \\ \hline C_1 & 0 & \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \\ C & I & \end{array} \right]$.

Therefore, if K_{FC} stabilizes G_{FC}, then $K_{OE} := \mathcal{F}_l(P_{OE}, K_{FC})$ stabilizes G_{OE}. Furthermore,

$$\mathcal{F}_l(G_{FC}, K_{FC}) = \mathcal{F}_l(\mathcal{S}(G_{OE}, P_{OE}), K_{FC}) = \mathcal{F}_l(G_{OE}, \mathcal{F}_l(P_{OE}, K_{FC})).$$

Thus, the class of equivalent stabilizing controllers can be characterized as

$$\mathcal{K}_{OE} = \{\mathcal{F}_l(P_{OE}, K_{FC}) : K_{FC} \in \mathcal{K}_{FC}\} = \{\mathcal{F}_l(J_{OE}, Q) : Q \in \mathcal{RH}_\infty\} \tag{5}$$

with

$$J_{OE} = \left[\begin{array}{c|cc} A - BC_1 + LC & L & -B \\ \hline C_1 & 0 & I \\ C & I & 0 \end{array} \right].$$

Since if $\begin{bmatrix} L \\ Q \end{bmatrix} \in \mathcal{K}_{FC}$ is given, then $\mathcal{F}_l(P_{OE}, \begin{bmatrix} L \\ Q \end{bmatrix}) = \mathcal{F}_l(J_{OE}, Q)$.

In fact, we have the following result [20]:

Proposition 2.4 *The OE controller set $\mathcal{K}_{OE}$ defined by (5) characterizes all stabilizing controllers for the OE system (4).*

2.3 Stabilizing Controller Parametrization

Consider system of G_L (1) with $D = 0$. Since (A, B) is stabilizable, there is a constant matrix F such that $A + BF$ is stable. Therefore, $\begin{bmatrix} F & 0 \end{bmatrix}$ is a special FI stabilizing controller. Now define a new system (by changing variables)

$$G_{tmp} = \left[\begin{array}{c|cc} A & B_1 & B \\ \hline -F & 0 & I \\ C & D_{21} & 0 \end{array}\right].$$

It is clear that a controller stabilizes G_L if and only if it stabilizes G_{tmp}. However, G_{tmp} is of the OE structure with $A + BF$ being stable, so the stabilizing controller can be obtained in terms of corresponding FC problem. Since (C, A) is assumed to be detectable, there exists a L such that $A + LC$ is stable. By Proposition 2.4, all controllers stabilizing G_{tmp} are given by $K = \mathcal{F}_l(J, Q)$, where

$$J = \left[\begin{array}{c|cc} A+BF+LC & L & -B \\ \hline -F & 0 & I \\ C & I & 0 \end{array}\right] = \left[\begin{array}{c|cc} A+BF+LC & -L & B \\ \hline F & 0 & I \\ -C & I & 0 \end{array}\right]. \tag{6}$$

Therefore, by Proposition 2.4 and the inversion formula for a linear fractional transformation [15, Lemma 2.7], we have the following statement.

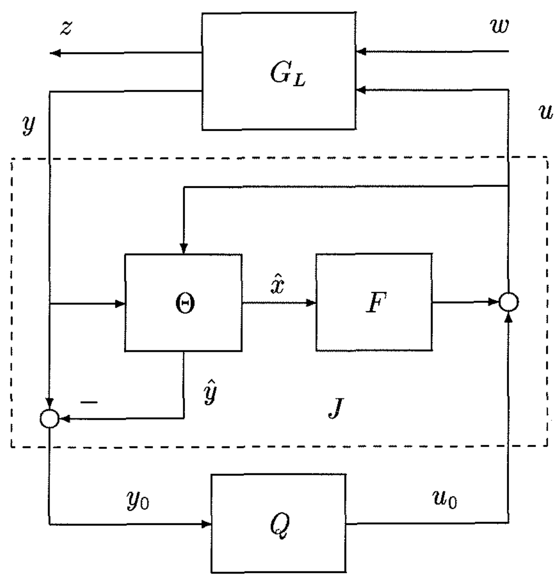
Theorem 2.5 *Consider the linear system G_L. Let F and L be such that $A + LC$ and $A + BF$ are stable, then all controllers stabilize G can be parametrized as $\mathcal{F}_l(J, Q)$, where J is given by (6), $Q \in \mathcal{RH}_\infty$. In addition, given any stabilizing controller K for G_L, it can be parametrized as $K = \mathcal{F}_l(J, Q)$ with $Q = \mathcal{F}_l(\hat{J}, K) \in \mathcal{RH}_\infty$, where*

$$\hat{J} = \left[\begin{array}{c|cc} A & L & B \\ \hline -F & 0 & I \\ C & I & 0 \end{array}\right] \tag{7}$$

It is observed that the central controller for this parametrization, i.e., the parametrized controller with the parameter $Q = 0$, is an observer-based controller, and the observer is as follows

$$\Theta : \begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \\ \hat{y} = C\hat{x} \end{cases}$$

where $\hat{x}$ is the estimated state x of the original system. Therefore, a parametrized controller has some separation structure, and the closed loop system is structured as follows



Next, consider the closed loop map which is also parametrized as

$$T_{zw} = \mathcal{F}_l(G, K) = \mathcal{F}_l(G, \mathcal{F}_l(J, Q)) =: \mathcal{F}_l(T, Q)$$

We have the following theorem about the structures of the closed loop maps, whose proof is straightforward and is omitted.

Theorem 2.6 *Let the state feedback and output injection matrices F abd L are chosen as in the previous theorem. Then the closed loop maps are parametrized as $T_{zw} = \mathcal{F}_l(T, Q)$ with*

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} := \left[\begin{array}{cc|cc} A+BF & -BF & B_1 & B \\ 0 & A+LC & B_1+LD_{21} & 0 \\ \hline C_1+D_{12}F & -D_{12}F & D_{11} & D_{12} \\ 0 & C & D_{21} & 0 \end{array}\right] \tag{8}$$

and $Q \in \mathcal{RH}_\infty$. Moreover, T_{zw} is affine in Q, i.e.,

$$T_{zw} = T_{11} + T_{12} Q T_{21} \tag{9}$$

3 Stabilization of Input-Affine Nonlinear Systems

From now on, we shall deal with nonlinear systems. We first consider a simple class of nonlinear systems, i.e., the input-affine systems, in this section; the results can be extended to handle a broader class of systems, and the extension is the main issue of the next section. Basically, the techniques to be used are demonstrated in the last section. In this section, it is examined to what extent the treatments and results for linear systems in the last section can be extended to handle the input-affine nonlinear systems. All results are local, unless otherwise noted.

3.1 Preliminaries: Stabilizability and Detectability

The reader is referred to [16] for the basic Lyapunov stability theory (see also the survey article [32]). Consider a nonlinear system with a input-affine realization:

$$G : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{10}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$ are the input and output vectors, respectively. We will assume $f, g, h \in \mathbf{C}^2$, and $f(0) = 0, h(0) = 0$. Therefore, $0 \in \mathbb{R}^n$ is an equilibrium of both systems with $u = 0$.

Definition 3.1 *(i) The dynamical system G (10) (or $[f(x), g(x)]$) is said to be locally smoothly (or exponentially) stabilizable if there is a $\mathbf{C}^2$ function $F : \mathbb{R}^n \to \mathbb{R}^p$ such that $\dot{x} = f(x) + g(x)F(x)$ is locally asymptotically (or exponentially) stable about $x = 0$.*

(ii) The dynamical system G (10) (or $[h(x), g(x)]$) is said to be (locally) smoothly (or exponentially) detectable if there is a $\mathbf{C}^2$ matrix-valued function $L : \mathbb{R}^n \to \mathbb{R}^{n \times q}$ such that $\dot{x} = f(x) + L(x)h(x)$ is (locally) asymptotically (or exponentially) stable about $x = 0$.

The global versions of stabilizability and detectability can be defined similarly. The definition of stabilizability is quite standard. The detectability notion is defined in terms of output injection, which is analogical to the one in the linear case. However, the output injection depends on the state variable. This consideration is just of technical interests, since the implication for the detectability notion is that if the system is locally exponentially detectable, then there exists a local state observer for the original system, and the observer can be constructed by the output injection.

Remark 3.1 *The smooth stabilizability and the smooth detectability can be characterized in terms of Lyapunov functions. For instance, from the inverse Lyapunov theorem, it follows that system G (10) is (locally) smoothly stabilizable, if and only if there are a (locally) $\mathbf{C}^3$ positive definite function (i.e., Lyapunov function for the closed loop system) $V : \mathbb{R}^n \to \mathbb{R}^+$, a smooth function $F : \mathbb{R}^n \to \mathbb{R}^p$ with $F(0) = 0$, and $\mathbf{C}^2$ functions $\gamma_1, \gamma_2, \gamma_3$ of class $\mathcal{K}$ such that:*

$$\gamma_1(\|x\|) \leq V(x) \leq \gamma_2(\|x\|) \tag{11}$$

$$\frac{\partial V(x)}{\partial x}(f(x) + g(x)F(x)) \leq -\gamma_3(\|x\|) \tag{12}$$

for $x \in \mathcal{B}_r$ with some $r > 0$.

Moreover, system (10) is locally exponentially stabilizable if and only if its linearized system around 0 is stabilizable, it is also noted that besides conditions (11) and (12), the necessary and sufficient conditions for local exponential stabilizability additionally require that

$$\lim_{s \to 0} \frac{\gamma_3(s)}{s^2} \in (0, \infty). \tag{13}$$

To conclude the review, we give a Hamilton-Jacobi Inequality (HJI) characterization about stabilizability and detectability.

Proposition 3.2 *Consider the system G (10).*

(i) It is (locally) smoothly stabilizable, if there exists a (locally) $\mathbf{C}^3$ positive definite function $V : \mathbb{R}^n \to \mathbb{R}^+$ with $\gamma_1(\|x\|) \leq V(x) \leq \gamma_2(\|x\|)$ for some $\mathbf{C}^2$ functions γ_1, γ_2 of class $\mathcal{K}$ such that the following HJI is satisfied for $x \in \mathcal{B}_r$ with some $r > 0$.

$$\frac{\partial V(x)}{\partial x} f(x) - \frac{\partial V(x)}{\partial x} g(x) g^T(x) \frac{\partial V^T(x)}{\partial x} \leq -\gamma_3(\|x\|) \tag{14}$$

with γ_3 being of class $\mathcal{K}$. Moreover, $F(x) = -g^T(x)\frac{\partial V^T(x)}{\partial x}$ is such a stabilizing state feedback controller.

(ii) It is (locally) smoothly detectable, if there exists a (locally) $\mathbf{C}^3$ *positive definite function* $U : \mathbb{R}^n \to \mathbb{R}^+$ *with* $\sigma_1(\|x\|) \le U(x) \le \sigma_2(\|x\|)$ *for some* $\mathbf{C}^2$ *functions* σ_1, σ_2 *of class* $\mathcal{K}$ *such that, for* $x \in \mathcal{B}_r$ *with some* $r > 0$*, the following HJI is satisfied.*

$$\frac{\partial U(x)}{\partial x} f(x) - h^T(x)h(x) \le -\sigma_3(\|x\|) \tag{15}$$

with σ_3 *being of class* $\mathcal{K}$*, and there is a* $\mathbf{C}^2$ *matrix-valued function* $L(x)$ *such that*

$$\frac{\partial U(x)}{\partial x} L(x) = -h^T(x). \tag{16}$$

Moreover, $u = L(x)y$ *is such a stabilizing output injection.*

The proof is straightforward, so it is omitted. It is noted that the above characterizations are just sufficient in general (, system $\dot{x} = x^4 + x^2 u$ is such a counter-example for stabilizability condition (14)). However, they are also necessary for linear systems. Moreover, we have the following result about exponentially stability and detectability.

Proposition 3.3 *Consider the system G (10).*

(i) It is locally exponentially stabilizable if and only if there exists a (locally) $\mathbf{C}^3$ *positive definite function* $V : \mathbb{R}^n \to \mathbb{R}^+$ *with* $\gamma_1(\|x\|) \le V(x) \le \gamma_2(\|x\|)$ *for some* $\mathbf{C}^2$ *functions* γ_1, γ_2 *of class* $\mathcal{K}$ *such that the HJI (14) is satisfied for* $x \in \mathcal{B}_r$ *with some* $r > 0$ *and* γ_3 *being of class* $\mathcal{K}$ *and satisfying* $\lim_{s \to 0} \frac{\gamma_3(s)}{s^2} \in (0, \infty)$*. Moreover,* $F(x) = -g^T(x)\frac{\partial V^T(x)}{\partial x}$ *is a locally exponentially stabilizing state feedback controller.*

(ii) It is locally exponentially detectable if and only if there are a (locally) $\mathbf{C}^3$ *positive definite function* $U : \mathbb{R}^n \to \mathbb{R}^+$ *with* $\sigma_1(\|x\|) \le U(x) \le \sigma_2(\|x\|)$ *for some* $\mathbf{C}^2$ *functions* σ_1, σ_2 *of class* $\mathcal{K}$ *such that, for* $x \in \mathcal{B}_r$ *with some* $r > 0$*, the HJI (15) is satisfied with* σ_3 *being of class* $\mathcal{K}$ *and* $\lim_{s \to 0} \frac{\sigma_3(s)}{s^2} \in (0, \infty)$*. Moreover,* $u = L(x)y$ *with* $L(x)$ *being a solution to (16) is such a locally exponentially stabilizing output injection.*

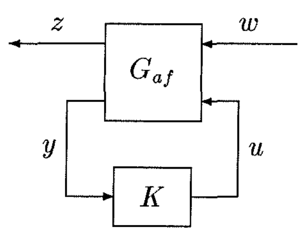
Proof The sufficiency is straightforward. The necessity follows from the fact that the linearized system of G is stabilizable and detectable. In fact, consider part (i), from [24], it follows that there exists a positive definite matrix P such that,

$$PA^T + AP - BB^T < 0$$

with $A = \frac{\partial f}{\partial x}(0), B = g(0)$. Now define $V(x) = x^T P^{-1} x$, it follows that V locally satisfies HJI (14) with some γ_3 being of class $\mathcal{K}$ and satisfying (13). Similar argument applies to part (ii). □

3.2 Controller Parametrization Problem Statement

In this section, the standard feedback configuration is as follows.



The plant has the following input-affine realization

$$G_{af}: \begin{cases} \dot{x} = f(x) + g_1(x)w + g(x)u \\ z = h_1(x) + k_{11}(x)w + k_{12}(x)u \\ y = h(x) + k_{21}(x)w + k(x)u \end{cases} \tag{17}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^p$ and $w \in \mathbb{R}^{p_1}$ are input vectors, and $y \in \mathbb{R}^q$ and $z \in \mathbb{R}^{q_1}$ are output vectors, respectively. We will assume $f, g_1, g, h_1, h, k_{ij}, k \in \mathbf{C}^2$, and $f(0) = 0, h_1(0) = 0, h(0) = 0$. Therefore, $0 \in \mathbb{R}^n$ is an equilibrium of the system with $w = 0$ and $u = 0$. It is known that a controller locally stabilizes system G_{af} with $w = 0$ if and only if it stabilizes the following system:

$$G: \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) + k(x)u \end{cases} \tag{18}$$

In the following discussion, although we could guess the controller parametrization formula from the linear case, prove it directly using Lyapunov theory under some additional conditions, and develop a theory without involving the input w and output z, the development would be less appealing for the following reasons: (i) The alternative approach provides a constructive proof; (ii) The techniques used in the linear case could fail some where, the parallel treatments to the linear case would reveal this; (iii) It is natural to take the extra input w and output z into account to reveal some I/O properties for nonlinear systems. For example, with this (I/O) consideration, we shall see that, unlike in the linear case, the parametrized I/O maps are not affine in the parameters.

In this section, we assume that the system G is *locally smoothly stabilizable and locally smoothly detectable.* We are interested in finding a time-invariant controller $u = Ky$ which has the following input-affine realization:

$$K: \begin{cases} \dot{\tilde{x}} = a(\tilde{x}) + b(\tilde{x})y \\ u = c(\tilde{x}) + d(\tilde{x})y \end{cases}$$

with $a, b, c, d \in \mathbf{C}^2$ and $a(0) = 0, c(0) = 0$, such that the closed loop system $\mathcal{F}_l(G_{af}, K)$ is asymptotically stable with $w = 0$. It is assumed that $I - k(x)d(\tilde{x})$ is invertible to guarantee the well posedness of the feedback system. We shall assume $k(x) = 0$ for simplicity.

We are interested in the following problem: **To what extent, can the parametrization formula for linear systems be extended to handle the input-affine nonlinear systems?** Actually, in this section, we shall parametrize a class of input-affine time invariant controllers which locally asymptotically stabilize G such that the parametrized controllers are characterized as fractional transformation of some locally stable parameters. The following definition gives a class $\mathcal{SP}_{af}$ of locally stable nonlinear time-invariant parameters:

Definition 3.2 *A class $\mathcal{SP}_{af}$ of input-affine nonlinear systems is so defined that each system in $\mathcal{SP}_{af}$ has a input-affine realization like (18) and is locally asymptotically stable around* 0 *with zero input.*

So if $Q \in \mathcal{SP}_{af}$, by inverse Lyapunov theorem, it admits a $\mathbf{C}^3$ Lyapunov function $V_Q(\cdot)$.

Definition 3.3 *Consider a system G, two controllers K and K' are equivalent if their corresponding closed loop map are identical for zero initial conditions, i.e. $\mathcal{F}_l(G, K) = \mathcal{F}_l(G, K')$, written as $K \cong K'$.*

3.3 State-Feedback and Output-Injection

As in the linear case, the construction of controller parametrization is accomplished by decomposing the original output feedback problem into some simpler problems, which are known as full information (FI) and full control (FC) problems. In this subsection, those problems are considered.

We first deal with full information (FI) system, in which case both state and disturbance w are measured.

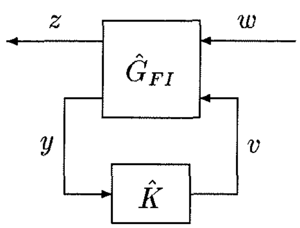
$$G_{FI}: \left\{ \begin{array}{rcl} \dot{x} &=& f(x) + g_1(x)w + g(x)u \\ z &=& h_1(x) + k_{11}(x)w + k_{12}(x)u \\ y &=& \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w \end{array} \right. \tag{19}$$

It is assumed that $[f(x), g(x)]$ is smoothly stabilizable, therefore, there exists a $\mathbf{C}^2$ function F such that $u = F(x)$ is a smooth stabilizing state feedback. Since both state x and disturbance w are available to the control input u, the control law $u = F(x) + Qw$ with $Q \in \mathcal{SP}_{af}$ is legal and it stabilizes the FI structure (19). Moreover, we have the following result about the parametrization of stabilizing controllers for FI structure.

Proposition 3.4 *Let $F : \mathbb{R}^n \to \mathbb{R}^p$ with $F(0) = 0$ be a smooth function such that $\dot{x} = f(x) + g(x)F(x)$ has an asymptotically stable equilibrium at $x = 0$. Then every input-affine stabilizing controller for FI structure (19) is equivalent to one of the controllers in the parametrized set as follows*

$$\mathcal{K}_{FI} = \left\{ \begin{bmatrix} F(\cdot) & Q \end{bmatrix} : Q \in \mathcal{SP}_{af} \right\}.$$

Proof It can be easily verified that the control law $u = F(x) + Qw$ with $Q \in \mathcal{SP}_{af}$ stabilizes the FI structure by Vidyasagar's stability theorem for cascade systems [40]. Now, given a stabilizing controller $\mathcal{K}_{FI}$, we need to show that there is a $Q \in \mathcal{SP}_{af}$ such that $K_{FI} \cong \begin{bmatrix} F(\cdot) & Q \end{bmatrix}$. To this end, make a change of control variable as $v = u - F(x)$, where x denotes the state of the system G_{FI}, then the feedback system with the controller K_{FI} has the following block diagram:



where

$$\hat{G}_{FI}: \left\{ \begin{array}{rcl} \dot{x} &=& f(x) + g(x)F(x) + g_1(x)w + g(x)v \\ z &=& h_1(x) + k_{12}(x)F(x) + k_{11}(x)w + k_{12}(x)v \\ y &=& \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w \end{array} \right.$$

$$\hat{K} = K_{FI} - \begin{bmatrix} F(\cdot) & 0 \end{bmatrix}.$$

Let Q be the map from w to v; it belongs to $\mathcal{SP}_{af}$ by asymptotic stability of the closed loop system. Then $u = F(x) + v = F(x) + Qw$. It follows that $\mathcal{F}_l(G_{FI}, K_{FI}) = \mathcal{F}_l(G_{FI}, \begin{bmatrix} F(\cdot) & Q \end{bmatrix})$ provided that the initial states are zero, so $K_{FI} \cong \begin{bmatrix} F(\cdot) & Q \end{bmatrix}$. $\square$

Remark 3.5 *If the system $[f(x), g(x)]$ is globally smoothly stabilizable, then there is a smooth $F_{I/O} : \mathbb{R}^n \to \mathbb{R}^p$, such that the system*

$$\dot{x} = f(x) + g(x)F_{I/O}(x) + g(x)v$$

with input v is I/S stable [29] (see Definition 4.4). Then by Sontag's theory [29] and the argument used in the above proof, it can be shown that every input-affine globally stabilizing controller is equivalent to $K_{FI} = F_{I/O}(x) + Qw$ with $Q \in \mathcal{SP}_{af}$ being globally asymptotically stable about 0.

As in the linear case, the stabilization problem where the control is directly injected to the state is considered next. Such structure is called full control (FC):

$$G_{FC} : \begin{cases} \dot{x} &= f(x) &+ g_1(x)w &+ \begin{bmatrix} I & 0 \end{bmatrix} u \\ z &= h_1(x) &+ k_{11}(x)w &+ \begin{bmatrix} 0 & I \end{bmatrix} u \\ y &= h(x) &+ k_{21}(x)w & \end{cases} \tag{20}$$

It is assumed that $[h(x), f(x)]$ is smoothly detectable. Thus, there exists a $\mathbf{C}^2$ smooth function $L : \mathbb{R}^n \to \mathbb{R}^{n\times q}$ such that $u = \begin{bmatrix} L(x) \\ 0 \end{bmatrix} y$ is a smooth stabilizing output injection. The following statement follows easily.

Proposition 3.6 *Let $L(\cdot)$ be a smooth matrix function such that $\dot{x} = f(x) + L(x)h(x)$ has an asymptotically stable equilibrium at $x = 0$. Then the following parametrized set characterizes a class of stabilizing controllers for FC structure (20).*

$$\mathcal{K}_{FC} = \left\{ \begin{bmatrix} L(x) \\ Q \end{bmatrix} : Q \in \mathcal{SP}_{af} \right\}.$$

It is noted that the controller is also allowed to depend on the state x. This consideration is mainly of technical interests as we will see soon.

Remark 3.7 *If the system $[h(x), f(x)]$ is globally smoothly detectable, let $L(\cdot)$ be a smooth matrix function such that $\dot{x} = f(x) + L(x)h(x)$ has a globally asymptotically stable equilibrium at $x = 0$. Then from Sontag's argument [29], it follows that $K_{FC} = \begin{bmatrix} L(x) \\ Q \end{bmatrix} y$ with $Q \in \mathcal{SP}_{af}$ being I/S stable globally stabilizes system G_{FC}.*

3.4 Locally Stabilizing Controller Parametrization

The main results of this section are given in this subsection. It will be shown that a class of input-affine (locally) stabilizing controllers are parametrized as fractional transformation of the parameters in $\mathcal{SP}_{af}$; the structures of the parametrized closed-loop maps are also examined. However, unlike linear systems, the closed loop maps are not affine in Q.

3.4.1 Controller Parametrization

We now consider the general output feedback stabilization problem. The solutions to this problem are based on the results in the last subsection. The nonlinear time-invariant plant is an input-affine system G_{af} (17) with $k_{22}(x) = 0$. It is assumed that $[f(x), g(x)]$ is locally smoothly stabilizable and $[h(x), f(x)]$ is locally smoothly detectable. So there are two $\mathbf{C}^3$ positive definite functions $V, U : \mathbb{R}^n \to \mathbb{R}^+$, two $\mathbf{C}^2$ functions $F : \mathbb{R}^n \to \mathbb{R}^p$, $L : \mathbb{R}^n \to \mathbb{R}^{n\times q}$, and two $\mathbf{C}^2$ functions γ and σ of class $\mathcal{K}$, such that:

$$\mathcal{H}_{SF}(V, F, x) := \frac{\partial V(x)}{\partial x}(f(x) + g(x)F(x)) \le -\gamma(\|x\|) \tag{21}$$

$$\mathcal{H}_{OI}(U, L, x) := \frac{\partial U(x)}{\partial x}(f(x) + L(x)h(x)) \le -\sigma(\|x\|) \tag{22}$$

for $x \in \mathcal{B}_r$ with some $r > 0$.

As in the linear case, make a change of variable, i.e., let $v = u - F(x)$, then we get the following system

$$G_{OE} : \left\{ \begin{array}{rcrcrcr} \dot{x} & = & f(x) & + & g_1(x)w & + & g(x)u \\ v & = & -F(x) & + & & + & u \\ y & = & h(x) & + & k_{21}(x)w & & \end{array} \right. \tag{23}$$

which has a constraint that $\dot{x} = f(x) + g(x)F(x)$ is asymptotically stable at $x = 0$. As far as the local asymptotic stabilization is concerned, $u = Ky$ stabilizes G_{OE} if and only if it stabilizes G_{af}. The above structure of G_{OE} is known as output estimation (OE).

Unlike the linear case, the two structures FC, which is discussed in the last subsection, and OE are not equivalent in the strict sense if it is just assumed that $[h(x), f(x)]$ is smoothly detectable. But there are indeed some close relations between this two structures. We can therefore take advantage of the FC results to deal with OE problem as in the linear case. Analogically, define a system P_{OE},

$$P_{OE} : \left\{ \begin{array}{rclcc} \dot{\tilde{x}} & = & f(\tilde{x}) + g(\tilde{x})F(\tilde{x}) & + & \begin{bmatrix} I & -g(\tilde{x}) \end{bmatrix} u_0 \\ u & = & -F(x) & + & \begin{bmatrix} 0 & I \end{bmatrix} u_0 \\ y_0 & = & -h(\tilde{x}) & + & y \end{array} \right.$$

Intuitively, motivated by the linear treatment, we would expect that $G_{FC} = \mathcal{S}(G_{OE}, P_{OE})$. However, this conjecture generally fails in this case, although the internal dynamics for both systems with zero inputs are identical if suitable initial conditions are chosen. As we only consider local stabilization, there arises a natural question: can we still use the FC local controllers to recover the OE controllers by $K_{OE} = \mathcal{F}_l(P_{OE}, K_{FC})$ as in the linear case? Or can the system $\mathcal{F}_l(G_{OE}, K_{OE})$ remain (locally) stable? The answer is positive if some stronger assumption about the detectability is made.

In this case, it is additionally assumed that $[h(x), f(x)]$ is locally exponentially detectable. Therefore, there are a $\mathbf{C}^3$ locally positive definite function $U : \mathbb{R}^n \to \mathbb{R}^+$, a locally smooth function $L : \mathbb{R}^n \to \mathbb{R}^p$, and a $\mathbf{C}^2$ function σ of class $\mathcal{K}$ such that (22) is satisfied, and in addition,

$$\lim_{s\to 0} \frac{\sigma_3(s)}{s^2} \in (0, \infty). \tag{24}$$

We have the following result about the stabilization, where the controller is recovered by $K = \mathcal{F}_l(P_{OE}, K_{FC})$ with $K_{FC} = \begin{bmatrix} L \\ Q \end{bmatrix}$.

Theorem 3.8 *Consider the system (17). Suppose that it is locally smoothly stabilizable and locally exponentially detectable. Let $F(\cdot)$ and $L(\cdot)$ be determined by the above characterizations (21), (22), and (24). Then the controller*

$$K : \begin{cases} \dot{\tilde{x}} = f(\tilde{x}) + g(\tilde{x})F(\tilde{x}) + L(\tilde{x})h(\tilde{x}) - L(\tilde{x})y \\ u = F(\tilde{x}) \end{cases} \tag{25}$$

(locally) asymptotically stabilizes system G_{af} around 0.

Moreover, the controller parametrized as $u = \mathcal{F}_l(M,Q)y$ with

$$M : \begin{cases} \dot{\tilde{x}} &= f(\tilde{x}) + g(\tilde{x})F(\tilde{x}) + L(\tilde{x})h(\tilde{x}) & - & L(\tilde{x})y & + & g(\tilde{x})u_0 \\ u &= F(\tilde{x}) & + & & & u_0 \\ y_0 &= -h(\tilde{x}) & + & y & & \end{cases} \tag{26}$$

for all $Q \in \mathcal{SP}_{af}$ also (locally) asymptotically stabilizes system G around 0.

We use Lyapunov technique to prove the above theorem. First, we have the following observation.

Lemma 3.9 *Let $U(x) \geq 0$ and $L(x)$ be taken from the above theorem, and let $x, \tilde{x}$ be states of systems G_{OE} and K_{OE}, $e = \tilde{x} - x$. Define*

$$\mathcal{H}_E(e,x) := \frac{\partial U(e)}{\partial e}(f(e+x) - f(x) + L(e+x)(h(e+x) - h(x))) + \frac{\partial U(e)}{\partial e}(g(x) - g(e+x))F(e+x).$$

Then for all $e, x \in \mathcal{B}_r$ with some $r > 0$, there exists a function π of class $\mathcal{K}$ with

$$\lim_{s \to 0} \frac{\pi(s)}{s^2} \in (0, \infty),$$

such that $\mathcal{H}_E(e,\tilde{x}) + \pi(\|e\|) \leq 0$.

Proof Recall that

$$\mathcal{H}_{OI}(U,L,e) := \frac{\partial U(e)}{\partial e}(f(e) + L(e)h(e)) \leq -\sigma(\|e\|).$$

for a $\mathbf{C}^2$ function σ of Class $\mathcal{K}$. The conclusion follows by observing that the Hessian matrix of $\mathcal{H}_E(e,\tilde{x})$ with respect to e at 0 can be arbitrarily close to the one of $\mathcal{H}_{OI}(U,L,e)$ with respect to e at 0 if $\tilde{x} \in \mathcal{B}_r$ for r small enough. In this case, the Hessian matrix of $\mathcal{H}_E(e,\tilde{x})$ is negative definite. The conclusion follows. □

The following lemma, which is from [31, Corollary 5.1], is used in the proof.

Lemma 3.10 *Suppose system $\dot{x} = f(x,u)$ with $f \in \mathbf{C}^0$ has an asymptotically stable equilibrium at 0 when $u = 0$. Then there exists a continuous function $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\sigma(s) \neq 0$ if $s \neq 0$, such that given a number $r > 0$ there is $r_m > 0$, for each $r_0 \in (0, r_m)$, if initial state $x(0) \in \mathcal{B}_{r_0}$ and $u \in \mathcal{L}_\infty[0,\infty)$ for which $\|u\|_\infty \leq \sigma(r_0)$ and $u(t) \to 0$ as $t \to \infty$, then the solution $x(t)$ exists with $\|x(t)\| < r$ for all $t \in \mathbb{R}^+$, and it satisfies $x(t) \to 0$ as $t \to \infty$.*

Next, we give a proof of the stabilization result which closely follows the treatments of Sontag in [31].

Proof **(Theorem 3.8)** Only the latter statement that $u = \mathcal{F}_l(M, Q)$ with $Q \in \mathcal{SP}_{af}$ locally stabilizes system (17) is proved, as the central controller (25) is obtained by letting $Q = 0$.

Consider $\mathcal{F}_l(G_{af}, \mathcal{F}_l(M, Q))$ for $Q \in \mathcal{SP}$ which has the following realization ($u_0 = Qy_0$):

$$\left\{\begin{array}{l} \dot{\xi} = a(\xi) + b(\xi)y_0 \\ u_0 = c(\xi) + d(\xi)y_0 \end{array}\right.$$

So the dynamics of closed loop system with $w = 0$ is as follows,

$$\left\{\begin{array}{l} \dot{x} = f(x) + g(x)F(\tilde{x}) + g(x)(c(\xi) + d(\xi)(-h(\tilde{x} + h(x))) \\ \dot{\tilde{x}} = f(\tilde{x}) + g(\tilde{x})F(\tilde{x}) + L(\tilde{x})(h(\tilde{x}) - h(x)) + g(\tilde{x})(c(\xi) + d(\xi)(-h(\tilde{x}) + h(x))) \\ \dot{\xi} = a(\xi) + b(\xi)(-h(\tilde{x}) + h(x)) \end{array}\right.$$

Let $e = \tilde{x} - x$, the reorganization of the system yields

$$\left\{\begin{array}{l} \dot{e} = \eta(e, x) + (g(e + x) - g(x))(c(\xi) + d(\xi)(-h(e + x + h(x))) \\ \dot{\xi} = a(\xi) + b(\xi)(-h(e + x) + h(x)) \\ \dot{x} = f(x) + g(x)F(e + x) + g(x)(c(\xi) + d(\xi)(-h(e + x + h(x))) \end{array}\right.$$

where η is a function defined as

$$\eta(e, x) := f(e + x) - f(x) + L(x + e)(e + h(x) - h(x)) + (g(e + x) - g(x))F(e + x)$$

The proof of stability of the closed-loop system is divided into the following three steps.

Step 1. We first prove that $e(t) \to 0$ as $t \to \infty$ if $e(0) \in \mathcal{B}_{r_0}$, $\xi \in \mathcal{B}_{r_q}$ and $x \in \mathcal{B}_r$, for some $r_0, r, r_q > 0$. Consider the e-subsystem

$$\dot{e} = \eta(e, x) + (g(e + x) - g(x))(c(\xi) + d(\xi)(-h(e + x + h(x))) \tag{27}$$

Take the U as given in the theorem, it was shown in Lemma 3.9 that there exists $r > 0$, for $x \in \mathcal{B}_r, e \in \mathcal{B}_r$,

$$\frac{\partial U(e)}{\partial e}\eta(e, x) = \mathcal{H}_E(e, x) \le -\pi(\|e\|)$$

where π is a function of of class $\mathcal{K}$ with $\lim_{s \to 0} \frac{\pi(s)}{s^2} \in (0, \infty)$. Therefore, there exists $r_q > 0$, and for all $\xi \in \mathcal{B}_{r_q}$, there is a function π_0 of class $\mathcal{K}$ such that for all $e, x \in \mathcal{B}_r$ with updated $r > 0$,

$$-\pi(\|e\|) + \frac{\partial U(e)}{\partial e}(g(e + x) - g(x))(c(\xi) + d(\xi)(-h(e + x + h(x))) \le -\pi_0(\|e\|).$$

Thus, for all $e, x \in \mathcal{B}_r$,

$$\begin{aligned} \dot{U}(e) &= \frac{\partial U(e)}{\partial e}\eta(e, t) + \frac{\partial U(e)}{\partial e}(g(e + x) - g(x))(c(\xi) + d(\xi)(-h(e + x + h(x))) \\ &\le -\pi(\|e\|) + \frac{\partial U(e)}{\partial e}(g(e + x) - g(x))(c(\xi) + d(\xi)(-h(e + x + h(x))) \\ &\le -\pi_0(\|e\|). \end{aligned}$$

Therefore, there is a function β_o of class $\mathcal{KL}$ such that

$$\|e(t)\| \le \beta_o(\|e(0)\|, t) \tag{28}$$

for all $t \in \mathbb{R}^+$, $e(0) \in \mathcal{B}_{r_0} \subset \mathbb{R}^n$ for some $r_0 > 0$ such that $\beta_o(r_0, 0) < r$, and $x \in \mathcal{B}_r, \xi \in \mathcal{B}_{r_q}$. Thus, $e(t) \to 0$ uniformly on $x \in \mathcal{B}_r, \xi \in \mathcal{B}_{r_q}$ as $t \to \infty$ if $e(0) \in \mathcal{B}_{r_0}$. Without loss of generality, it is assumed that r and r_q are chosen such that (28) holds for all $t \in \mathbb{R}^+$, $e(0) \in \mathcal{B}_{r_0}$, and $x \in \bar{\mathcal{B}}_r, \xi \in \bar{\mathcal{B}}_{r_q}$, where $\bar{\mathcal{B}}_r$ is the closure of $\mathcal{B}_r$.

Step 2. We will next show $\xi(t) \to 0$ as $t \to \infty$ for $\xi(0) \in \mathcal{B}_{q_0}$, $x(0) \in \mathcal{B}_{r_0}$, and $x(t) \in \mathcal{B}_r$ for some $q_0, r_0, r > 0$. Consider the ξ-subsystem

$$\dot{\xi} = a(\xi) + b(\xi)(-h(e + x) + h(x)) \tag{29}$$

(with e as an input). If $e = 0$, then the system becomes $\dot{\xi} = a(\xi)$ which is locally stable. By Lemma 3.10, there exists a continuous function σ with $\sigma(s) \neq 0$ if $s \neq 0$, for the given $r_q > 0$, there is $q_0 > 0$ with $q_0 < r_q$ such that if $\|e(t)\| < \sigma(q_0)$ with $e(t) \to 0$ as $t \to \infty$, then $\|\xi(t)\|_\infty < r_q$ and $x(t) \to 0$ as $t \to \infty$. Therefore, if the above conditions for $e(t)$ satisfying (28) are satisfied, then it is done.

We now verify that $e(t)$ satisfying (28) has the required properties for some suitable $r_0 > 0$. We first show that $\xi(t)$ satisfying (29) also satisfies $\|\xi(t)\|_\infty < r_q$ for all $\xi(0) \in \mathcal{B}_{q_0}$, and e evolving according to (27) with $e(0) \in \mathcal{B}_{r_0}$ for some $r_0 > 0$ and $x \in \mathcal{B}_r$. In fact, r_0 is adjusted such that $\beta_o(r_0, 0) \leq \sigma(q_0)$. Suppose there is a time T such that $\|\xi(T)\| \geq r_q$ with some $\xi(0) \in \mathcal{B}_{q_0}$ (here T is chosen to be the minimal such time). As $\|\xi(t)\| \leq r_q$ for $t \in [0, T]$, then for all $x(t) \in \mathcal{B}_r$, (28) is satisfied for $t \in [0, T]$, i.e., $\|e(t)\| \leq \beta_o(\|e(0)\|, t) \leq \beta_o(\|e(0)\|, 0) < \sigma(q_0)$ for $t \in [0, T]$. By the previous statement and the causality of system (29), we have $\xi(t) \in \mathcal{B}_{r_q}$ for $t \in [0, T]$ which contradicts the assumption $\|\xi(T)\| \geq r_q$. Therefore, $\xi(t) \in \mathcal{B}_{r_q}$ for $t \in [0, \infty)$. Thence, if $e(0) \in \mathcal{B}_{r_0}$ and $\|x\|_\infty < r$ then $\|e(t)\| \leq \beta_o(\|e(0)\|, t) \leq \beta_o(\|e(0)\|, 0) < \sigma(q_0)$ for $t \in [0, \infty)$. Therefore, by Lemma 3.10, it is concluded that $\xi(t) \to 0$ as $t \to \infty$ for $\xi(0) \in \mathcal{B}_{q_0}$, $x(0) \in \mathcal{B}_{r_0}$, and $x(t) \in \mathcal{B}_r$ for some $q_0, r_0, r > 0$.

Step 3. Finally, we prove $x(t) \to 0$ if $x(0), e(0) \in \mathcal{B}_{r_0}$ and $\xi(0) \in q_0$. Consider the x-subsystem

$$\dot{x} = f(x) + g(x)F(e + x) + g(x)(c(\xi) + d(\xi)(-h(e + x + h(x))) \tag{30}$$

Note that if $(e, \xi) = 0$, then the system becomes $\dot{x} = f(x) + g(x)F(x)$ which is asymptotically stable by assumption. It is also known from the above proofs that if $\xi(0) \in \mathcal{B}_{q_0}$, $x(0) \in \mathcal{B}_{r_0}$, and $x(t) \in \mathcal{B}_r$ for some $q_0, r_0, r > 0$, then $e(t) \to 0$ and $\xi(t) \to 0$ as $t \to \infty$. By the same argument as in Step 2, it is shown that $\|x(t)\| < r$ and $x(t) \to 0$ if $x(0), e(0) \in \mathcal{B}_{r_0}$ and $\xi(0) \in q_0$ for some suitably adjusted $r_0, q_0 > 0$.

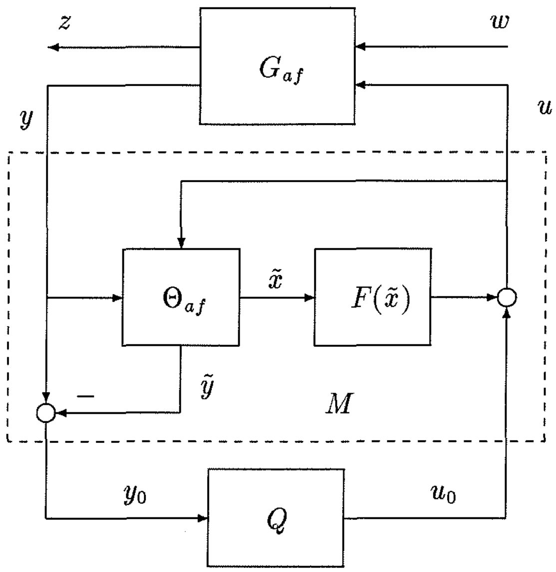
This completes the proof. $\square$

3.4.2 Separation Structures of Parametrized Controllers

The parametrized controller has a separation structure, and it is an observer-based controller. The observer is as follows.

$$\Theta_{af} : \begin{cases} \dot{\tilde{x}} = f(\tilde{x}) + g(\tilde{x})F(\tilde{x}) + L(\tilde{x})(h(\tilde{x}) - y) \\ \tilde{y} = h(\tilde{x}) \end{cases}$$

The estimated state is $\tilde{x}$ and $\tilde{x}(t) - x(t) \to 0$ as $t \to \infty$ for $w = 0$ because of the locally exponentially stability. The feedback system with a parametrized controller is thus structured as the following diagram.



3.5 Structures of Closed-Loop Maps

In this subsection, we will consider structures of the closed maps with the parametrized controllers. We first have the following definition of an I/O property.

Definition 3.4 *Consider an I/O operator $P : \mathcal{L}^e_\infty[0,\infty) \to \mathcal{L}^e_\infty[0,\infty)$. It is said to be locally I/O stable if there are $k_I, k_O > 0$ such that for all $w \in \mathcal{L}_\infty[0,\infty)$ with $\|w\|_\infty \le k_I$, then $z := Pw \in \mathcal{L}_\infty[0,\infty)$ and $\|z\|_\infty \le k_O$.*

We first have the following lemma about the relation between asymptotic stability and I/O stability for a nonlinear system, which follows from [31, Corollary 5.1] (see also [38, Lemma 4.1]).

Lemma 3.11 *Consider the following system*

$$G : \begin{cases} \dot{x} = f_c(x) + g_c(x)w \\ z = h_c(x) + k_c(x)w \end{cases}$$

with $f_c, g_c, h_c, k_c \in \mathbf{C}^0$. It is assumed that $\dot{x} = f_c(x)$ is locally asymptotically stable around 0. Then given $\epsilon > 0$, there is a $\delta > 0$, such that for all $\|w\|_\infty \le \delta$, $\|z\|_\infty \le \epsilon$.

Next, consider the closed-loop map from w to z which is parametrized as follows,

$$T_{zw} = \mathcal{F}_l(G_{af}, \mathcal{F}_l(M, Q)) = \mathcal{F}_l(T, Q), \quad Q \in \mathcal{SP}_{af}$$

where T has the following realization.

$$T : \begin{cases} \dot{x} &=& f(x) + g(x)F(\tilde{x}) &+& g_1(x)w &+& g(x)u_0 \\ \dot{\tilde{x}} &=& f(\tilde{x}) + g(\tilde{x})F(\tilde{x}) + L(\tilde{x})(h(\tilde{x}) - h(x)) &-& L(\tilde{x})k_{21}(x)w &+& g(\tilde{x})u_0 \\ z &=& h_1(x) + k_{12}(x)F(\tilde{x}) &+& k_{11}(x)w &+& k_{12}(x)u_0 \\ y_0 &=& h(\tilde{x}) - h(x) &+& k_{21}(x)w & & \end{cases} \tag{31}$$

Now consider the structure of I/O map T which has zero initial conditions $x(0) = 0$ and $\tilde{x}(0) = 0$, the closed-loop map T_{zw} is locally I/O stable and parametrized as follows:

$$T_{zw}w = T_1(w, T_2(Q)w)$$

where T_1 and $T_2(Q)$ are locally I/O maps, and T_1 defines the map from $\begin{bmatrix} w \\ u_0 \end{bmatrix}$ to z as follows,

$$T_1 : \begin{cases} \dot{x} &= f(x) + g(x)F(\tilde{x}) &+& g_1(x)w &+& g(x)u_0 \\ \dot{\tilde{x}} &= f(\tilde{x}) + g(\tilde{x})F(\tilde{x}) + L(\tilde{x})(h(\tilde{x}) - h(x)) &-& L(\tilde{x})k_{21}(x)w &+& g(\tilde{x})u_0 \\ z &= h_1(x) + k_{12}(x)F(\tilde{x}) &+& k_{11}(x)w &+& k_{12}(x)u_0 \end{cases}$$

and $T_2(Q)$ is the map from w to u_0 defined as follows.

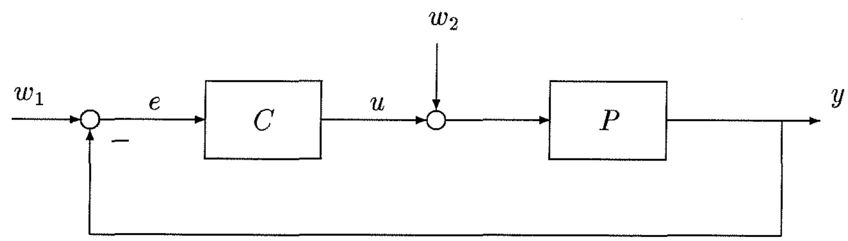
$$T_2(Q) : \begin{cases} \dot{x} = f(x) + g(x)F(\tilde{x}) + g_1(x)w + g(x)Q(h(\tilde{x}) - h(x) + k_{21}(x)w) \\ \dot{\tilde{x}} = f(\tilde{x}) + g(\tilde{x})F(\tilde{x}) + L(\tilde{x})(h(\tilde{x}) - h(x) - k_{21}(x)w) + g(\tilde{x})Q(h(\tilde{x}) - h(x) + k_{21}(x)w) \\ u_0 = Q(h(\tilde{x}) - h(x) + k_{21}(x)w) \end{cases}$$

Both T_1 and $T_2(Q)$ are locally asymptotically stable with zero inputs as guaranteed from the development, then they are locally I/O stable by Lemma 3.11.

It is noted that that unlike in the linear case, **the closed loop maps have no affine-like relation with Q for nonlinear systems in general.** To conclude the I/O discussion, we consider an example from [8]. The parametrization of the closed-loop map is used in the nonlinear $\mathcal{H}_\infty$-optimal controller design in [12].

An Example

Consider a feedback system with the following block diagram,

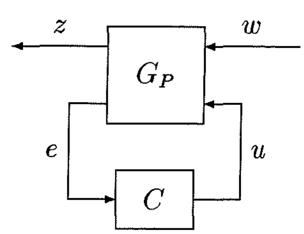


P is a (locally) I/O stable plant. we need to parametrize a class of controllers C such that the resulting maps from $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} u \\ y \end{bmatrix}$ are (locally) I/O stable. This problem is considered in [8] in an I/O setting. In the following, we consider it in the state space framework.

Suppose system P has the following input-affine realization.

$$P : \begin{cases} \dot{x} = f_P(x) + g_P(x)w_0 \\ z = h_P(x) \end{cases}$$

with $f_P, g_P, h_P \in \mathbf{C}^2$; and $\dot{x} = f_P(x)$ **is locally exponentially stable around** 0. Define $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and $z = \begin{bmatrix} u \\ y \end{bmatrix}$, then the system block diagram is redrawn as follows.



with

$$G_P : \begin{cases} \dot{x} &= f_P(x) &+ \begin{bmatrix} 0 & g_P(x) \end{bmatrix} w &+ g_P(x)u \\ z &= \begin{bmatrix} 0 \\ h_P(x) \end{bmatrix} &+ & \begin{bmatrix} I \\ 0 \end{bmatrix} u \\ e &= -h_P(x) &+ \begin{bmatrix} I & 0 \end{bmatrix} w & \end{cases}$$

As P is assumed to be locally exponentially stable around 0, the state feedback and output injection can be chosen as $F = 0$ and $L = 0$, respectively. Then by Theorem 3.8, a class of controllers C can be represented as $C = \mathcal{F}_l(M_P, Q)$, with

$$M_P : \begin{cases} \dot{\tilde{x}} = f_P(\tilde{x}) + g_P(\tilde{x})u_0 \\ u = u_0 \\ y_0 = h_P(\tilde{x}) + e \end{cases}$$

with $Q \in \mathcal{SP}_{af}$. Next, we examine the structure of the parametrized controller.

Lemma 3.12 *The controller $C = \mathcal{F}_l(M_P, Q)$ has structure $C = Q(I - PQ)^{-1}$.*

Proof let $u_0 = Qy_0$, then

$$C : \begin{cases} \dot{\tilde{x}} = f_P(\tilde{x}) + g_P(\tilde{x})Q(h_P(\tilde{x}) + e) \\ u = Q(h_P(\tilde{x}) + e) \end{cases}$$

with $\tilde{x}(0) = 0$. Let $y_0 = h_P(\tilde{x}) + e$. Since $u = Qy_0$, it is sufficient to show $y_0 = (I - PQ)^{-1}e$, or $e = (I - PQ)y_0$.

In fact, consider $y_u := (I - PQ)y_0$, it can be written as

$$\begin{cases} \dot{x} = f_P(x) + g_P(x)Qy_0 \\ y_u = y_0 - h_P(x) \end{cases}$$

Now replace $y_0 := h_P(\tilde{x}) + e$, then we have

$$\begin{cases} \dot{x} = f_P(x) + g_P(x)Q(h_P(\tilde{x}) + e) \\ y_u = h_P(\tilde{x}) + e - h_P(x) \end{cases}$$

Now as $x(0) = 0$, by the uniqueness of the solution to differential equations, we have $x(t) = \tilde{x}(t)$. Then $y_u = e$, i.e., $e = (I - PQ)y_0$. □

We finally examine the structure of the closed loop map from w_1 to y provided $w_2 = 0$.

Lemma 3.13 *The closed loop map from w_1 to y with $w_2 = 0$ is parametrized as $T_{zw_1} = PQ$.*

Proof Let $w_2 = 0$. then the closed loop map is parametrized as $T_{zw_1} = \mathcal{F}_l(T, Q)$ where T is given in (31):

$$T : \begin{cases} \dot{x} = f_P(x) + g_P(x)u_0 \\ \dot{\tilde{x}} = f_P(\tilde{x}) + g_P(\tilde{x})u_0 \\ z = h_P(x) \\ y_0 = h_P(\tilde{x}) - h_P(x) + w \end{cases}$$

$x(0) = \tilde{x}(0) = 0$, and $u_0 = Qy_0$. Therefore,

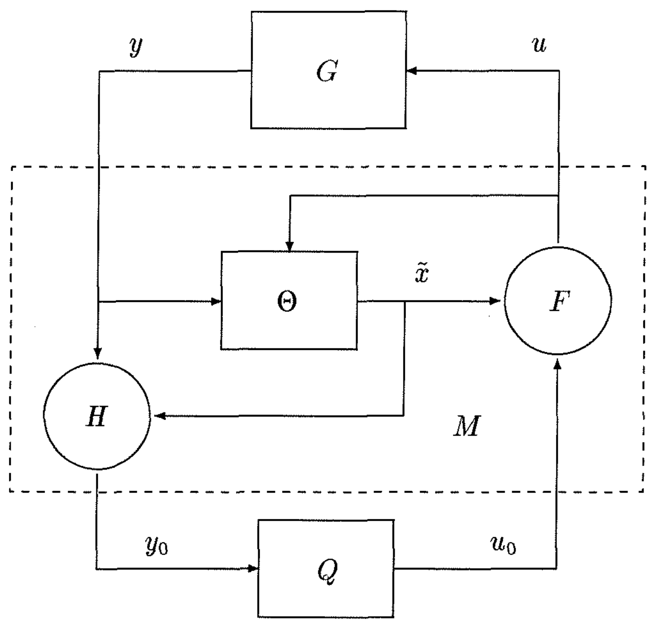
$$T_{zw_1} : \begin{cases} \dot{x} = f_P(x) + g_P(x)Q(h_P(\tilde{x}) - h_P(\tilde{x}) + w) \\ \dot{\tilde{x}} = f_P(\tilde{x}) + g_P(\tilde{x})Q(h_P(\tilde{x}) - h_P(x) + w) \\ z = h_P(x) \end{cases}$$

with $x(0) = \tilde{x}(0) = 0$. Therefore, $\tilde{x}(t) = x(t)$ for all $t \geq 0$, then $T_{zw_1} = PQ$ which is locally I/O stable. □

The interested reader can compare the above results with those in [8].

4 Stabilization of General Nonlinear Systems

In the last section, we have considered the input-affine nonlinear systems, which have nice structures close to linear systems. The stabilizing control laws and stabilizing controller parametrizations are constructed based on observers. In this section, we consider the stabilizing controller parametrizations of a more general class of nonlinear systems whose structures are not required to be input-affine. It will be verified that a set of stabilizing controllers for system G will be characterized as fractional transformations of some stabilizing parameters, i.e., $\mathcal{F}_l(M, Q)$. Motivated by the previous results on input-affine systems, the parametrized controllers have a separation structure. Therefore, the starting point for further consideration is that there must exist observers for the concerned class of systems. Therefore, the feedback systems will have the following structure, where Θ denotes the observer.



4.1 Local Controller Parametrization

The plant considered in this subsection is

$$G_g : \begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases} \tag{32}$$

where $f(0,0) = 0, h(0) = 0$, $f, h \in \mathbf{C}^0$; x, u and y are assumed to have dimensions n, p, and q, respectively. Clearly, the origin 0 is an equilibrium of the system with $u = 0$. In this subsection, the locally stabilizing controller parametrization for system G_g (32) is considered. The parametrized controllers are represented as fractional transformations of some locally stable parameters. We first define the following a class of the local stable parameters.

Definition 4.1 *The class $\mathcal{SP}_{loc}$ of time-invariant nonlinear systems is so defined that each member has the following realization.*

$$Q : \left\{ \begin{array}{l} \dot{x} = f_Q(x,u) \\ y = h_Q(x,u) \end{array} \right. \tag{33}$$

for some $f_Q, h_Q \in \mathbf{C}^0$, and is locally asymptotically stable at 0 with $u = 0$.

So if $Q \in \mathcal{SP}_{loc}$, by inverse Lyapunov theorem, it admits a locally $\mathbf{C}^1$ Lyapunov function $V_Q(\cdot)$.

Next, the notions of stabilizability and detectability for system G_g (32) are examine.

Definition 4.2 *G_g (32) is locally stabilizable around $x = 0$ if there is a continuous function F : $\mathbb{R}^n \rightarrow \mathbb{R}^p$ with $F(0) = 0$ such that $\dot{x} = f(x, F(x))$ is locally asymptotically stable around $x = 0$.*

The following technical definition is from [39].

Definition 4.3 *System G_g is said to be locally weakly detectable, if there are a $\mathbf{C}^0$ mapping f_o : $\mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ with $f_o(0,0,0) = 0$, a $\mathbf{C}^1$ locally positive definite function $W : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^+$, and functions ϕ_1, ϕ_2, ϕ_3 of class $\mathcal{K}$ such that*

$$\phi_1(\|x - \tilde{x}\|) \leq W(x, \tilde{x}) \leq \phi_2(\|x - \tilde{x}\|) \tag{34}$$

$$\frac{\partial W(x,\tilde{x})}{\partial x} f(x,u) + \frac{\partial W(x,\tilde{x})}{\partial \tilde{x}} f_o(\tilde{x}, h(x), u) \leq -\phi_3(\|x - \tilde{x}\|) \tag{35}$$

for all $u \in \mathcal{B}_{r_u}$ and $x, \tilde{x} \in \mathcal{B}_r$ for some $r_u, r > 0$.

If the system G_g is (locally) weakly detectable, standard arguments show that there is a function β_0 of class $\mathcal{KL}$ such that the error state $e = \tilde{x} - x$ evolves according to the following dynamics.

$$\dot{e} = f_o(e + x, h(x), u) - f(x,u) =: \eta(e, x, u) \tag{36}$$

satisfies

$$\|e(t)\| \leq \beta_0(\|e(0)\|, t) \tag{37}$$

for all $t \in \mathbb{R}^+$, $x \in \mathcal{B}_r$, and $u \in \mathcal{B}_{r_u}$. Therefore, $\tilde{x} \rightarrow x$ as $t \rightarrow \infty$, i.e., the system $\dot{\tilde{x}} = f_o(\tilde{x}, h(x), u)$ is a local observer for system G_g.

Remark 4.1 *A local observer for the general system (32) with $f, h \in \mathbf{C}^2$ can be constructed if there is a matrix-valued function $L(x)$ such that $\dot{x} = f(x,0) + L(x)h(x)$ is locally exponentially stable. In fact, it can be shown that $\dot{x} = f_o(\tilde{x}, y, u)$ with*

$$f_o(\tilde{x}, y, u) = f(\tilde{x}, u) + L(\tilde{x})(h(\tilde{x}) - y)$$

is such a local observer. It is exactly the case for the observers constructed for the input-affine systems in the last section.

We have the following theorem about the local controller parametrization for system (32).

Theorem 4.2 *Suppose system G_g (32) is locally asymptotically stabilizable and locally weakly detectable. If in addition, there is a $\mathbf{C}^0$ function $F : \mathbb{R}^n \to \mathbb{R}^p$ such that $\dot{x} = f(x, F(x))$ is locally asymptotically stable at 0, and the function $f_o : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^n$ is chosen as in the definition 4.3, then the controller parametrized as $u = \mathcal{F}_l(M_l, Q)y$ with M_l given by*

$$M_l : \begin{cases} \dot{\tilde{x}} = f_o(\tilde{x}, y, F(\tilde{x}) + u_0) \\ u = F(\tilde{x}) + u_0 \\ y_0 = h(\tilde{x}) - y \end{cases}$$

for all $Q \in \mathcal{SP}_{loc}$ also locally asymptotically stabilizes system G_g around 0.

The following proof basically follows the proof of Theorem 3.8, we just give a sketch here.

Proof Assume $u_0 = Qy_0$ for $Q \in \mathcal{SP}_{loc}$ with the following realization

$$\begin{cases} \dot{\xi} = a(\xi, y_0) \\ u_0 = c(\xi, y_0) \end{cases}$$

The dynamics of the closed loop system is described by

$$\begin{cases} \dot{x} = f(x, F(\tilde{x}) + c(\xi, h(\tilde{x}) - h(x))) \\ \dot{\tilde{x}} = f_o(\tilde{x}, h(x), F(\tilde{x}) + c(\xi, h(\tilde{x}) - h(x))) \\ \dot{\xi} = a(\xi, h(\tilde{x}) - h(x)) \end{cases}$$

Take $e := \tilde{x} - x$ as the error state, then equivalently, the closed loop system can be represented with the state $x_c = [e^T \ \xi^T \ x^T]^T$ as

$$\begin{cases} \dot{e} = f_o(e + x, h(x), u) - f(x, u) \\ \dot{\xi} = a(\xi, h(x + e) - h(x)) \\ \dot{x} = f(x, F(x + e) + c(\xi, h(x + e) - h(x))) \end{cases}$$

where $u = F(x + e) + c(\xi, h(x + e) - h(x))$.

We first prove that $e(t) \to 0$ and $\xi(t) \to 0$ as $t \to \infty$ if $(e(0), \xi(0)) \in \mathcal{B}_{r_0} \times \mathcal{B}_{q_0}$ for some $r_0, q_0 > 0$.

Consider the e-subsystem. By the detectability, there is a function β_o of class $\mathcal{KL}$ such that

$$\|e(t)\| \leq \beta_o(\|e(0)\|, t) \tag{38}$$

for all $t \in \mathbb{R}^+$, $e(0) \in \mathcal{B}_{r_0} \subset \mathbb{R}^n$ and $\beta_o(r_0, 0) < r$, $x \in \mathcal{B}_r \subset \mathbb{R}^n$, and $u \in \mathcal{B}_{r_u}$ for some $r_0, r, r_u > 0$. So $e(t) \to 0$ as $t \to \infty$, for all $e(0) \in \mathcal{B}_{r_0}$. As $u = F(x+e) + c(\xi, h(x+e) - h(x))$ is continuous function of e, ξ, and x, it can be assumed that $u \in \mathcal{B}_{r_u}$ if $e, x \in \mathcal{B}_r$ and $\xi \in \mathcal{B}_{r_q}$ for some $r_q > 0$.

Next, consider the ξ-subsystem $\dot{\xi} = a(\xi, h(x + e) - h(x))$. If $e = 0$, then it becomes $\dot{\xi} = a(\xi, 0)$ which is locally asymptotically stable. By Lemma 3.10, there exists a continuous function σ with $\sigma(s) \neq 0$ if $s \neq 0$, for the given $r_q > 0$, there is $q_0 > 0$ with $q_0 < r_q$ such that if $\|e(t)\| < \sigma(q_0)$ with $e(t) \to 0$ as $t \to \infty$, then $\|\xi(t)\| < r_q$ and $x(t) \to 0$ as $t \to \infty$. However, from the similar arguments as in the proof of Theorem 3.8 (Step 2), it follows that $e(t)$, which satisfies (38) also satisfies the above conditions for $\xi(0) \in \mathcal{B}_{q_0}$, $e(0) \in \mathcal{B}_{r_0}$, and $x(t) \in \mathcal{B}_r$ for some $q_0, r_0, r > 0$. Therefore, by Lemma 3.10, it is concluded that $\xi(t) \to 0$ as $t \to \infty$.

The proof of this theorem is completed by showing that $x(t) \to 0$ as $t \to \infty$ if $x(0), e(0) \in \mathcal{B}_{r_0}$ and $\xi(0) \in q_0$ for some suitably adjusted $r_0, q_0 > 0$, but the latter follows the similar arguments as above.

$\square$

Remark 4.3 *The central controller for this parametrization can be recovered by letting $Q = 0$. By doing so, we have Vidyasagar's Theorem [39] as follows:*

Suppose that system G_g is locally asymptotically stabilizable and locally weakly detectable. If $\mathbf{C}^0$ function $F : \mathbb{R}^n \to \mathbb{R}^p$ is such that $\dot{x} = f(x, F(x))$ is locally asymptotically stable at 0, and $\mathbf{C}^0$ function $f_o : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^n$ is chosen as in the definition 4.3, then the controller $u = Ky$ given by K:

$$\left\{ \begin{array}{l} \dot{\tilde{x}} = f_o(\tilde{x}, y, F(\tilde{x})) \\ u = F(\tilde{x}) \end{array} \right.$$

locally asymptotically stabilizing the feedback system at $0 \in \mathbb{R}^n \times \mathbb{R}^n$.

4.2 Global Controller Parametrization

In this subsection, we generalize the local result in the previous subsection to get a global characterization. However, the conditions in general are very restrictive. The system considered is the same in (32).

$$G_g : \left\{ \begin{array}{l} \dot{x} = f(x, u) \\ y = h(x) \end{array} \right. \tag{39}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$ are the input and output vectors, respectively. $f, h \in \mathbf{C}^0$, $f(0,0) = 0, h(0) = 0$. Clearly, the origin 0 is an equilibrium of the system with $u = 0$. It is assumed that for all $u \in \mathcal{L}_\infty^e[0, \infty)$, $x(t)$ is defined for all initial state $x(0) \in \mathbb{R}^n$ and (almost) all $t \in \mathbb{R}^+$. We first have the following technical definition due to Sontag [29].

Definition 4.4 *Consider system $\dot{x} = f(x, u)$. It is input-to-state (I/S) stable if there exist functions β of class $\mathcal{KL}$ and γ of class $\mathcal{K}$ such that for each essentially bounded measurable control $u(\cdot)$ and each initial state $x(0)$, the solution $x(t)$ exists for each $t \geq 0$; and furthermore, it satisfies*

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|u\|_\infty) \tag{40}$$

Definition 4.5 *The class $\mathcal{SP}_{I/S}$ of nonlinear systems is defined such that each system in $\mathcal{SP}_{I/S}$ has a realization like (33) and is I/S stable.*

The following definition is due to Vidyasagar [39].

Definition 4.6 *System G_g (39) is said to be globally weakly detectable, if there are a $\mathbf{C}^0$ mapping $f_o : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^n$ with $f_o(0,0,0) = 0$, a $\mathbf{C}^1$ positive definite function $W : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$, and functions ϕ_1, ϕ_2, ϕ_3 of class $\mathcal{K}_\infty$ such that*

$$\phi_1(\|x - \tilde{x}\|) \leq W(x, \tilde{x}) \leq \phi_2(\|x - \tilde{x}\|) \tag{41}$$

$$\frac{\partial W(x, \tilde{x})}{\partial x} f(x, u) + \frac{\partial W(x, \tilde{x})}{\partial \tilde{x}} f_o(\tilde{x}, h(x, u), u) \leq -\phi_3(\|x - \tilde{x}\|) \tag{42}$$

for all $u \in \mathbb{R}^p$ and $x, \tilde{x} \in \mathbb{R}^n$.

By Lyapunov Theorem, the above definition of detectability implies that there is a function β_0 of class $\mathcal{KL}$ such that the error state $e := \tilde{x} - x$ satisfies

$$\|e(t)\| \leq \beta_0(\|e(0)\|, t) \tag{43}$$

for all $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, and $u \in R^p$. Therefore, system $\dot{\tilde{x}} = f_o(\tilde{x}, h(x,u), u)$ is an observer for system G.

Motivated by the construction of the stabilizing controller parametrization in the last subsection, we have the following result regarding the parametrization of a class of stabilizing controllers.

Theorem 4.4 *Suppose the system G_g (39) is globally stabilizable and globally weakly detectable. Let the function $f_o : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^n$ be chosen as in the definition 4.6. If in addition, there are a $\mathbf{C}^0$ function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ for some integer $m > 0$ such that $\dot{x} = f(x, F(x+v, w))$ with input $\begin{bmatrix} v \\ w \end{bmatrix}$ is I/S stable, and a $\mathbf{C}^0$ function $H : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^l$ for some integer $l > 0$ such that $\|H(x_1, h(x_2))\| \leq \kappa_h(\|x_1 - x_2\|)$ for some function κ_h of class $\mathcal{K}$ and for all $x_1, x_2 \in \mathbb{R}^n$. Then the controller parametrized as $u = \mathcal{F}_l(M_g, Q)y$ with M_g given by*

$$M_g : \begin{cases} \dot{\tilde{x}} = f_o(\tilde{x}, y, F(\tilde{x}, u_0)) \\ u = F(\tilde{x}, u_0) \\ y_0 = H(\tilde{x}, y) \end{cases}$$

for all $Q \in \mathcal{SP}_{I/S}$ also globally asymptotically stabilizes system G_g around 0.

Remark 4.5 *The central controller can be recovered by letting $Q = 0$. This theorem is reduced to the following statement about global stabilizability due to [33] (see also [18, 36]).*

Suppose the system G_g is globally asymptotically stabilizable and globally weakly detectable. Let the function $f_o : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^n$ be chosen as in the definition 4.6. If in addition, A $\mathbf{C}^0$ function $F : \mathbb{R}^n \to \mathbb{R}^p$ is such that $\dot{x} = f(x, F(x+v))$ with input v is I/S stable, then the controller $u = Ky$ given by

$$K : \begin{cases} \dot{\tilde{x}} = f_o(\tilde{x}, y, F(\tilde{x})) \\ u = F(\tilde{x}) \end{cases}$$

globally asymptotically stabilizes the feedback system at $0 \in \mathbb{R}^n \times \mathbb{R}^n$.

The above theorem is given in [22] in the case where the output y depends on both x and u, i.e., $y = h(x,u)$. In the following, we just give a sketch of the proof for completeness, and mainly emphasize on the discussion about the restrictiveness of the conditions.

Proof (Theorem 4.4: A Sketch) The proof basically follows Sontag's arguments about global stability of cascade systems. Assume $u_0 = Qy_0$ for $Q \in \mathcal{SP}_{I/S}$ with the following realization,

$$\begin{cases} \dot{\xi} = a(\xi, y_0) \\ u_0 = c(\xi, y_0) \end{cases}$$

The dynamics of the closed loop system is represented with the state $x_c = [e^T \ \xi^T \ x^T]^T$ (with $e := \tilde{x} - x$) as

$$\begin{cases} \dot{e} = f_o(e + x, h(x), u) - f(x,u) = \eta(e, x, u) \\ \dot{\xi} = a(\xi, H(x+e, h(x))) \\ \dot{x} = f(x, F(x+e, c(\xi, H(x+e, h(x))))) \end{cases}$$

with $u = F(x+e, c(\xi, H(x+e, h(x))))$, where $x, \tilde{x}, \xi$ are states of plant, observer, and the parameter, respectively.

By the detectability and the choice of f_o, there is a function β_o of class $\mathcal{KL}$ such that

$$\|e(t)\| \leq \beta_o(\|e(0)\|, t) \tag{44}$$

for all $t \in \mathbb{R}^+$, $e(0) \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, and $u \in R^p$.

Consider the ξ-system, the I/S stability of $\dot{\xi} = a(\xi, y_0)$ with input $y_0 = H(x+e, h(x))$ and the growth condition on H imply that there exist a function β_q of class $\mathcal{KL}$ and a function γ_h of class $\mathcal{K}$ such that

$$\|\xi(t)\| \leq \beta_q(\|\xi(0)\|, t) + \gamma_h(\|e\|_\infty) \tag{45}$$

for all $t \in \mathbb{R}^+$, $x(0) \in \mathbb{R}^n$, and essentially bounded function $e : \mathbb{R}^+ \rightarrow \mathbb{R}^n$.

Now according to Sontag's argument in [31, 18], define

$$\beta_1(s,t) = \beta_q(\beta_q(s, t/2) + \gamma_h(\beta_o(s,0), t/2) + \gamma_h(\beta_o(s, t/2) + \beta_o(s,t).$$

Then it is easy to verify that β_1 is also of class $\mathcal{KL}$. Define $\hat{x} = \begin{bmatrix} e \\ \xi \end{bmatrix}$, from (44) and (45), it follows

$$\|\hat{x}(t)\| \leq \beta_1(\|\hat{x}(0)\|, t), \forall t \in \mathbb{R}^+.$$

Finally, consider the x-system $\dot{x} = f(x, F(x+e, u_0))$ with $u_0 = c(\xi, H(x+e, h(x)))$. The I/S stability assumption about the system with $\begin{bmatrix} e \\ u_0 \end{bmatrix}$ as the input, the continuity of function c, and the growth condition on H together imply that there exist a function β_s of class $\mathcal{KL}$ and a function γ_κ of class $\mathcal{K}$ such that

$$\|x(t)\| \leq \beta_s(\|x(0)\|, t) + \gamma_\kappa(\|\hat{x}\|_\infty)$$

Now use the similar argument as the above, we can conclude that there is a function β of class $\mathcal{KL}$ such that

$$\|x_c(t)\| \leq \beta(\|x_c(0)\|, t), \forall t \in \mathbb{R}^+.$$

($x_c = \begin{bmatrix} \hat{x} \\ x \end{bmatrix}$). This concludes the globally asymptotic stability of the closed loop system. □

Further Remarks

In general, the conditions in Theorem 4.4, i.e. the I/S stability condition and the growth rate condition on H, are restrictive. In the following, we will examine some examples that satisfy the conditions.

Remark 4.6 *A class of smoothly stabilizable (input-affine) nonlinear systems satisfy the I/S stability conditions in Theorem 4.4 [33, 18, 22]. More generally, let's consider a globally (smoothly) stabilizable nonlinear system,* $\dot{x} = f(x,u)$ *with* f *smooth. By Sontag's arguments [31, 18], there exists a feedback law* $u = F_{I/O}(x)$ *and a smooth function* $G : \mathbb{R}^n \to \mathbb{R}^+$ *with* $0 < G(x) < C_m < \infty$ *for all* $x \in \mathbb{R}^n$ *such that the following system*

$$\dot{x} = f(x, F_{I/O}(x) + G(x)v)$$

with input v *is I/S stable, i.e., there are a function* β *of class* $\mathcal{KL}$ *and a function* γ_0 *of class* $\mathcal{K}$ *such that*

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma_0(\|v\|_\infty)$$

for all $x(0) \in \mathbb{R}^n$ *and* $t \in \mathbb{R}^+$. *In addition, assume* $F_{I/O}$ *is a globally Lipschitz function with constant* $C > 0$, *and* $G(x) \geq C_g$ *with some* $C_g > 0$ *for all* $x \in \mathbb{R}^n$. *Define function* $F : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ *as* $F(x,w) = F_{I/O}(x) + w$. *Therefore, for* $x(t)$ *which satisfies*

$$\dot{x} = f(x, F(x+v, w)) = f(x, F_{I/O}(x) + (F_{I/O}(x+v) - F_{I/O}(x) + w)),$$

we have

$$\begin{aligned}
\|x(t)\| &\leq \beta(\|x(0)\|, t) + \gamma_0(\|G^{-1}(x)(F_{I/O}(x+v) - F_{I/O}(x) + w)\|_\infty) \\
&\leq \beta(\|x(0)\|, t) + \gamma_0(\frac{1}{C_g}(C(\|v\|_\infty) + \|w\|_\infty)) \\
&\leq \beta(\|x(0)\|, t) + \gamma(\left\| \begin{bmatrix} v \\ w \end{bmatrix} \right\|_\infty)
\end{aligned}$$

with $\gamma(s) := \gamma_0(\frac{1+C}{C_g}s)$ *being of class* $\mathcal{K}$, *which implies that the system* $\dot{x} = f(x, F(x+v, w))$ *with input* $\begin{bmatrix} v \\ w \end{bmatrix}$ *is I/S stable.*

Remark 4.7 *A class of feedback linearizable input-affine nonlinear systems satisfy the I/S stability conditions in Theorem 4.4 [14]. Suppose* $\dot{x} = f(x) + g(x)u$ *with* $x \in \mathbb{R}^n, u \in \mathbb{R}$ *and* f, g *smooth is exactly linearizable, i.e., there is a coordinate transformation* $z = \Phi(x)$ *(in fact,* $z_i = L_f^{i-1}(x)$*) with* $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ *being a diffeomorphism such that the system under the new coordinate* $\dot{z} = \hat{f}(z) + \hat{g}(z)u$ *is as follows*

$$\begin{cases} \dot{z}_1 = z_2 \\ \cdots \\ \dot{z}_{n-1} = z_n \\ \dot{z}_n = b(z) + a(z)u \end{cases}$$

Then by [14, Theorem 1], there is a control law $u = \mu(z)$, *such that* $\dot{z} = \hat{f}(z) + \hat{g}(z)\mu(z)$ *is globally asymptotically stable and in addition,* $\dot{z} = \hat{f}(z) + \hat{g}(z)(\mu(z+d) + w)$ *with input* $\begin{bmatrix} d \\ w \end{bmatrix}$ *is I/S stable, i.e., there are a function* β *of class* $\mathcal{KL}$ *and a function* γ_0 *of class* $\mathcal{K}$ *such that*

$$\|z(t)\| \leq \beta(\|z(0)\|, t) + \gamma(\left\| \begin{bmatrix} d \\ w \end{bmatrix} \right\|_\infty)$$

for all $z(0) \in \mathbb{R}^n$ *and* $t \in \mathbb{R}^+$. *Now return to the original system, define* $F : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ $(p = 1)$ *as* $F(x, w) = \mu(\Phi(x)) + w$; *the original system becomes* $\dot{x} = f(x) + g(x)F(x + v, w)$ *and under the transformation* $z(t) = \Phi(x(t))$ *it becomes*

$$\dot{z} = \hat{f}(z) + \hat{g}(z)(\mu(z + ((\Phi(x + v) - \Phi(x))) + w)$$

As $\Phi^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ *is still a diffeomorphism, it is continuous, therefore there is a function* κ *of class* $\mathcal{K}$ *such that* $\|\Phi^{-1}(z)\| \leq \kappa(\|z\|)$. *Now we additionally assume* $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ *is globally Lipschitz with constant* $C > 0$, *then*

$$\begin{aligned}
\|x(t)\| = \|\Phi^{-1}(z(t))\| &\leq \kappa(\|z(t)\|) \leq \kappa(\beta(\|\Phi(x(0))\|, t) + \gamma(\left\| \begin{bmatrix} \Phi(x + v) - \Phi(x) \\ w \end{bmatrix} \right\|_\infty)) \\
&\leq \kappa(\beta(C\|x(0)\|, t) + \gamma(C\|v\|_\infty + \|w\|_\infty)) \\
&\leq \beta_c(\|x(0)\|, t) + \gamma_c(\left\| \begin{bmatrix} v \\ w \end{bmatrix} \right\|_\infty)
\end{aligned}$$

where $\beta_c(s, t) = \kappa(2\beta(Cs, t))$ *is again of class* $\mathcal{KL}$ *and* $\gamma_c(s) = \kappa(2\gamma(Cs + s))$ *is of class* $\mathcal{K}$.

Remark 4.8 *If the output function* h *is globally Lipschitz, then an* H, *which satisfies the requirement in the above theorem, can be taken as* $H(x, y) = h(x) - y$. *In fact,* $\|H(x_1, h(x_2))\| = \|h(x_1) - h(x_2)\| \leq C\|x_1 - x_2\|$ *for some* $C > 0$, *and* $\kappa_h(s) = Cs$ *is of class* $\mathcal{K}$.

From the above discussions, we see that the I/S stability conditions and growth conditions on H for the classes of systems we examined are reduced to the uniform continuity (or globally Lipschitz) conditions for some functions (as a reviewer pointed out). The global Lipschitz condition is a restrictive one.

5 Concluding Remarks

We have proposed a state-space approach to the parametrization of stabilizing controllers for time-invariant nonlinear systems without adopting coprime factorization technique. The central idea here is the decomposition of output feedback problem into simpler state feedback and state estimation problems. The stabilizing controllers are represented as fractional transformations of some stable parameters. Both local and global parametrizations are derived for the general nonlinear systems. These problems are treated under the assumption that the controllers have the same dimensions as the plants and have separation structures, and the observers are assumed to exist. However, in the general case, especially in the global case, the constructions of the observers are not provided.

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